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Lie extended symmetries and relativistic particles

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Abstract. Relativistic wave equations describing non-zero rest mass and spin 0, $\frac{1}{2}$ or 1 particles are studied with respect to first-order symmetries that they admit in the context of the Lie extended method. Their explicit forms are understood as elements of the Poincaré enveloping algebra in terms of well-known 4-vectors and second-order tensors. Superstructures in the Dirac case and parasuperstructures in the Kemmer case are pointed out. The generalization to arbitrary spins is also considered.

1. Introduction

Kinematic symmetries of *free* relativistic wave equations for arbitrary spin particles have already been obtained in Wigner's work [1] on the Poincaré group and its irreducible unitary representations [1, 2]. The corresponding results are quoted in different textbooks [3–6] and play a fundamental role, especially in particle physics [3] and in quantum field theory [4]. They correspond to the crucial notion of *Poincaré invariance* associated with inhomogeneous Lorentz coordinate transformations in Minkowski spacetime: they lead to 10 symmetries easily obtained from the classical infinitesimal Lie method [5–7], giving the well-known closed structure known as the 'Poincaré (Lie) algebra' generated by spacetime translations (P_μ, $\mu = 0, 1, 2, 3$) and rotations ($M_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, $M_{\mu\nu} = -M_{\nu\mu}$).

As far as non-zero mass particles of spin-$\frac{1}{2}$ are concerned, i.e. those which are described by the Dirac equation [8], new symmetries have recently been added (see [6, 9, 10] and references therein) but by considering the whole set of operators containing only first-order derivatives, without any demand concerning a closed structure. These new symmetries (which are complementary to the 10 Poincaré ones) have been collected as Lie extended symmetries (see [6] in particular) and there are 15 new ones besides the trivial unit operator, so that it makes sense to speak now about $10+15+1=26$ symmetries of the Dirac equation. We have especially reconsidered this study [10] but with specific purposes such as going to the limit case of zero rest mass particles like the Weyl neutrino, etc. Let us also recall that to these Lie extended symmetries correspond new conservation laws which cannot be found in the previous approach.

From another point of view, these Dirac symmetries can also generate some Lie superalgebras [11] as already noticed [9, 10]. The largest superstructures are in fact obtained when we add to the above extended symmetries the 10 second-order operators $P_\mu P_\nu$ as will be recalled later. Moreover, we have recently pointed out [12] that these

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Dirac extended symmetries contain *hidden* supersymmetries [13] generating, for example, a superalgebra sqm(3) in this free context. Such remarks strengthen the association between the Dirac theory and supersymmetric quantum mechanics very often exploited in recent works [14, 15].

Let us now ask if the above results are only true for spin-$\frac{1}{2}$ particles or if they can be appropriately extended to other spin contexts. Effectively, we plan to consider the case of free Kemmer particles [16, 17] and to study their Lie extended symmetries in connection with parasupersymmetric quantum mechanics [18, 19]. This association with the spin-1 case of the Kemmer theory has very recently been exploited in the construction of pararelativistic harmonic oscillators [20].

The contents of this paper are then as follows. In section 2 we show that the Lie extended symmetries of the Dirac equation can effectively be deduced from those of the Klein–Gordon equation. We first construct the invertible operators V and W and show how Dirac and Klein–Gordon problems can be intimately connected (section 2.1); the symmetries of the Dirac context are then simply recovered according to previous results [9] discussed [10] elsewhere. We also point out a new invariance superalgebra (section 2.2). In section 3 the Kemmer theory is considered in a parallel way, and the information given in section 3.1 leads, in the spin-1 case, to new explicit forms of the 15 Lie (extra) extended symmetries given in terms of Kemmer matrices. In section 3.2 we explicitly construct a Lie parasuperalgebra according to our mathematical developments, this Lie parasuperalgebra being expressed in terms of a Lie product given, for example, by double commutators [21]. The spin-0 case is then considered in section 3.3 by referring to a six-dimensional representation where the β_5 matrix takes a non-trivial form. Section 4 contains general comments and conclusions, mainly discussing the generators in terms of scalars, 4-vectors and second-order (symmetric or antisymmetric) tensors admitting only first-order derivatives as required. In that regard, the Bargmann–Wigner [22] and spin tensor operators [23, 24] play a fundamental role.

2. Extended symmetries of Dirac particles

Since the Wigner study of the irreducible unitary representations of the Poincaré group [1], we know that the Dirac relativistic equation admits 10 Lie symmetries including translations (P_μ) and Lorentz transformations ($M_{\mu\nu}$) such as pure rotations (M_{ij}) and boosts (M_{0i}). Other symmetries have recently been added: they are called 'extended Lie symmetries' and lead here to 15 non-trivial *first-order* generators [6, 9, 10]. With the 10 Poincaré symmetries besides the identity, we thus deal with 26 generators in the spin-$\frac{1}{2}$ case. They take the following forms for μ, $\nu = 0, 1, 2, 3$:

$$M_{\mu\nu} = L_{\mu\nu} + S^{\mathrm{D}}_{\mu\nu} \qquad L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \qquad S^{\mathrm{D}}_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu] \tag{2.1a}$$

$$P_\mu = \mathrm{i}\partial_\mu = \mathrm{i}\frac{\partial}{\partial x^\mu} \tag{2.1b}$$

$$W^{\mathrm{D}}_\mu = \tfrac{1}{2}\gamma_5(P_\mu - m\gamma_\mu) \tag{2.2a}$$

$$W^{\mathrm{D}}_{\mu\nu} = \tfrac{1}{2}\gamma_5(\gamma_\mu P_\nu - \gamma_\nu P_\mu) \tag{2.2b}$$

$$A^{\mathrm{D}}_\mu = \gamma_\mu + g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma} x_\nu W^{\mathrm{D}}_{\rho\sigma} \tag{2.2c}$$

$$B^{\mathrm{D}} = \tfrac{3}{2}\mathrm{i}\gamma_5 + ig^{\lambda\rho} x_\rho W^{\mathrm{D}}_\lambda \tag{2.2d}$$

where

$$\varepsilon_{0123} = -\varepsilon^{0123} = 1 \qquad \text{and} \qquad \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \tag{2.3a}$$

with [8, 10]

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4 \qquad \{\gamma^5, \gamma^\mu\} = 0. \tag{2.3b}$$

Here the Dirac theory is covariantly described by the equation [8]:

$$L_{\mathrm{D}}\Psi_{\mathrm{D}} \equiv (\mathrm{i}\gamma^\mu\partial_\mu - m)\Psi_{\mathrm{D}} = 0. \tag{2.4}$$

The results (2.1) and (2.2) are readily obtained by solving the usual requirement that

$$[Q_{\mathrm{D}}, L_{\mathrm{D}}] = \lambda L_{\mathrm{D}} \tag{2.5}$$

given that Q_{D} is a symmetry generator of equation (2.4), so that $Q_{\mathrm{D}}\Psi$ is once again a solution iff Ψ is a solution [6, 9, 10]. Let us insist that, in general, λ is an operator.

In order to extend these Dirac results to other spin cases, we will first construct two interesting operators V_{D} and W_{D} and show that they lead to a one-to-one correspondence between symmetries of the Dirac type (2.5) and those associated with the Klein–Gordon equation (section 2.1). Finally, we will then point out in this Dirac case a new invariance superalgebra (section 2.2).

2.1. *The invertible operators V_{D} and W_{D}*

It is well known that *each* component of the wavefunctions describing a *relativistic* system has to satisfy the Klein–Gordon equation. As is the case for the four ($=2(2s+1)$ when $s=\frac{1}{2}$) components of the Dirac wavefunction, we can search for information on the Dirac symmetries through those obtained (more easily) in the Klein–Gordon context. This leads to another economical method for getting the explicit results (2.1) and (2.2). It deals with the construction of the two operators V_{D} and W_{D} [25] leading to the property that equation (2.4) and

$$L'_{\mathrm{D}}\Psi'_{\mathrm{D}} = 0 \qquad L'_{\mathrm{D}} = W_{\mathrm{D}}L_{\mathrm{D}}V_{\mathrm{D}}^{-1} \qquad \Psi'_{\mathrm{D}} = V_{\mathrm{D}}\Psi_{\mathrm{D}} \tag{2.6a}$$

admit the *same* numbers of symmetries Q_{D} and Q'_{D}, respectively, where

$$Q'_{\mathrm{D}} = V_{\mathrm{D}}Q_{\mathrm{D}}V_{\mathrm{D}}^{-1}. \tag{2.6b}$$

Indeed, if Q is a symmetry operator of the Dirac equation, i.e. if it satisfies equation (2.5), then we have

$$[Q'_{\mathrm{D}}, L'_{\mathrm{D}}] = \lambda' L'_{\mathrm{D}} \tag{2.7a}$$

where

$$\lambda' = W_{\mathrm{D}}\lambda W_{\mathrm{D}}^{-1} - W_{\mathrm{D}}Q_{\mathrm{D}}W_{\mathrm{D}}^{-1} + V_{\mathrm{D}}Q_{\mathrm{D}}V_{\mathrm{D}}^{-1}. \tag{2.7b}$$

Conversely, if Q'_{D} satisfies equation (2.7a), it is easily proved that equation (2.5) holds.

Let us construct the following invertible operators:

$$V_{\mathrm{D}}^{\pm 1} = 1 \mp \frac{1}{2m}(1-\gamma_5)\gamma_\mu P^\mu \tag{2.8a}$$

and

$$W_{\mathrm{D}}^{\pm 1} = 1 \pm \frac{1}{2m}(1+\gamma_5)\gamma_\mu P^\mu. \tag{2.8b}$$

Due to the fact that V_D is not unitary, we can only conclude that there is a one-to-one correspondence given by equation (2.6*b*) between the numbers of symmetry operators Q_D and Q'_D although formulae (2.4) and (2.6*a*) are not equivalent.

We immediately get

$$L'_D = W^{+1} L_D V^{-1} = \frac{1}{2m}(1+\gamma_5)P_\mu P^\mu - m \tag{2.9}$$

so that, in the chiral Dirac representation characterized by a diagonal γ_5 matrix ($\gamma_5 = \mathrm{diag}(1,1,-1,-1)$), we learn that the Dirac equation (2.4) is now in correspondence with equations (2.6*a*) and (2.9), i.e. with the system

$$(P_\mu P^\mu - m^2)\Psi'_i = 0 \qquad (i=1,2) \qquad \Psi'_3 = \Psi'_4 = 0. \tag{2.10}$$

We consequently notice that we obtain ($(2s+1)$ for $s=\frac{1}{2}$) Klein–Gordon equations, each equation admitting the usual Poincaré symmetries associated with translations (P_μ) and Lorentz rotations ($M_{\mu\nu} \equiv L_{\mu\nu}$). It has to be pointed out that, through equation (2.6*b*), we have thus to consider the 10 transformed operators

$$V_D^{-1} L_{\mu\nu} V_D \qquad \text{and} \qquad V_D^{-1} P_\mu V_D. \tag{2.11}$$

Explicit calculations show that we recover the 10 expected Poincaré symmetries (2.1) but supplemented by linear combinations of *some* new ones appearing in equations (2.2). In order to get the *whole* set of first-order symmetries, i.e. the 26 symmetries generated by equations (2.1) and (2.2), we have to consider products of the above symmetries which evidently are still symmetries of the Dirac equation. Such interesting results are immediately understood by noticing the following identifications inside the 15 Dirac operators (2.2) acting on solutions of equation (2.4):

$$W_\mu^D = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \qquad M^{\nu\rho} = L^{\nu\rho} + S_D^{\nu\rho} \tag{2.12a}$$

$$W_{\mu\nu}^D = \frac{1}{m}(P_\mu W_\nu^D - P_\nu W_\mu^D) \tag{2.12b}$$

$$A_\mu^D = \frac{1}{m}(\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} W_D^\sigma - \tfrac{1}{2}P_\mu) \tag{2.12c}$$

$$B^D = \tfrac{1}{4}\varepsilon_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma} \tag{2.12d}$$

where the $M_{\mu\nu}$ and P_μ generators are the usual Dirac ones (equations (2.1)). The 4-vector (2.12*a*) is simply the Bargmann–Wigner operator [22] leading to the Pauli–Lubanski (Casimir) operator [26] classifying the irreducible unitary representations of the Poincaré group. Let us also note that the six components (2.12*b*) are characteristic of antisymmetric spin tensors [23, 24] of the Hilgevoord–Wouthuysen type while the time component of the 4-vector (2.12*c*) is a Dirac constant of motion which has already played an interesting role in quantum mechanics [27]. All the operators (2.12) belong to the Poincaré enveloping algebra generated by the usual $M_{\mu\nu}$ and P_μ operators: they are the only first-order operators. The arbitrary *n*th order has been considered elsewhere [28].

2.2. *The largest invariance superalgebra*

Due to the system (2.10) including the Klein–Gordon operator, let us introduce the second-order translation generators $P_\mu P_\nu$ in addition to the 26 Dirac symmetries.

Among them we can select the 31 operators

$$\{I, P_\mu, M_{\mu\nu}, P_\mu P_\nu, W^{\mathrm{D}}_\mu, W^{\mathrm{D}}_{\mu\nu}\} \tag{2.13}$$

and point out that they generate a Lie superalgebra [11]. There are 21 even operators $\{I, P_\mu, M_{\mu\nu}, P_\mu P_\nu\}$ satisfying commutation relations between themselves and 10 odd ones $\{W^{\mathrm{D}}_\mu, W^{\mathrm{D}}_{\mu\nu}\}$ satisfying anticommutation relations between themselves and commutation relations with the preceding even operators. This superalgebra has recently been used to show [12] that Dirac theory contains in particular a hidden supersymmetry in the free context as well as in some interacting cases. It also contains another previously noted [10] superalgebra having the structure

$$I \oplus [(P_\mu, M_{\mu\nu}) \square (P_\mu P_\nu, W^{\mathrm{D}}_{\mu\nu})] \tag{2.14}$$

where direct ($\oplus$) and semidirect ($\square$) sums appear.

3. Extended symmetries of Kemmer particles

Among the irreducible unitary representations of the Poincaré group [1], there are those associated with physical particles of non-zero rest mass with spin 1 or 0, the so-called vector or scalar mesons, respectively. These representations admit as a relativistic equation invariant under the Poincaré group the so-called Kemmer equation [16, 17]

$$L_{\mathrm{K}}\Psi_{\mathrm{K}} \equiv (\mathrm{i}\beta_\mu \partial^\mu - m)\Psi_{\mathrm{K}} = 0 \tag{3.1}$$

as derived by Kemmer, Duffin and Petiau. The four matrices $\beta_\mu (\mu = 0, 1, 2, 3)$ satisfy the characteristic structure relations of a K(4) Kemmer algebra

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\lambda\nu}\beta_\mu \tag{3.2}$$

which contains 126 linearly independent elements. Three of them belong to the centre and lead to the decomposition

$$126 = (1)^2 + (5)^2 + (10)^2 \tag{3.3}$$

in terms of dimensions of the irreducible representations. It is easy to convince ourselves that the usual Klein-Gordon equation describing free scalar mesons can be rewritten in the form (3.1) by dealing with 5×5 matrices satisfying the structure relations (3.2). The same is true for the relativistic description of vector mesons but by dealing now with 10×10 matrices satisfying the algebra (3.2). Each of these Kemmer descriptions evidently contains redundant components, three in the spin-0 case and four in the spin-1 case, ensuring $2\,(2s+1)$ independent components in each context as expected from a relativistic point of view. Let us also point out that the Kemmer equation (3.1) admits, as a covariant equation, the 10 Poincaré symmetries (2.1) associated with translations and spacetime rotations but with

$$S^{\mathrm{K}}_{\mu\nu} = \mathrm{i}[\beta_\mu, \beta_\nu] \tag{3.4}$$

replacing $S^{\mathrm{D}}_{\mu\nu}$ in equation (2.1a) in the Dirac context. Here the new problem corresponding to equation (2.5) is too complex, necessitating solution of the associated system and developing all matrices in the 126-dimensional basis of K(4). Consequently, we will use the other above-mentioned method, constructing two operators hereafter called V_{K} and W_{K} for evident reasons and showing that they lead once again to a useful one-to-one correspondence between the Kemmer and Klein–Gordon symmetries (section 3.1). Moreover, on the basis of the associations between Dirac theory and Lie superalgebras [14, 15] as well as between Kemmer theory and Lie parasuperalgebras [4, 20, 21, 29, 30], we will show in this Kemmer context the existence of a Lie parasuperalgebra generated by some of these Kemmer symmetries (section 3.2). Finally, we will give some details on the spin-0 case where specific difficulties can be circumvented (section 3.3).

3.1. Invertible operators V_K and W_K

In this Kemmer context, equation (3.1) and

$$L'_{\mathrm{K}}\Psi'_{K}=0 \qquad L'_{\mathrm{K}}=W_{\mathrm{K}}L_{\mathrm{K}}V_{\mathrm{K}}^{-1} \qquad \Psi'_{\mathrm{K}}=V_{\mathrm{K}}\Psi_{\mathrm{K}} \tag{3.5}$$

admit the same numbers of symmetries Q_{K} and Q'_{K}, respectively, where

$$Q'_{\mathrm{K}}=V_{\mathrm{K}}Q_{\mathrm{K}}V_{\mathrm{K}}^{-1}. \tag{3.6}$$

The argument given here is completely parallel to that previously developed in the Dirac context (see equations (2.6) and (2.7)). Let us then construct the operators

$$V_{\mathrm{K}}=1-\frac{1}{m}\beta_\mu P^\mu\beta_5^2+\frac{1}{2m^2}(\beta_5^2-\beta_5)[P_\mu P^\mu-2(\beta_\mu P^\mu)^2] \tag{3.7a}$$

$$V_{\mathrm{K}}^{-1}=1+\frac{1}{m}\beta_\mu P^\mu\beta_5^2-\frac{1}{2m^2}\left[P_\mu P^\mu\left(1-\frac{1}{m}\beta_\nu P^\nu\right)-2(\beta_\mu P^\mu)^2\right][\beta_5^2+\beta_5) \tag{3.7b}$$

and

$$W_{\mathrm{K}}^{\pm 1}=1\pm\frac{1}{m}\beta_5^2\beta_\mu P^\mu\mp\frac{1}{m^2}(\beta_\mu P^\mu)^2\beta_5. \tag{3.8}$$

Here β_5 is defined by

$$\beta_5=\frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma \tag{3.9}$$

and satisfies the relations (3.2). The resulting operator L'_{K} (see equations (3.5)) is then obtained as

$$L'_{\mathrm{K}}=\frac{1}{2m}(\beta_5^2+\beta_5)(P_\mu P^\mu-m^2)+\tfrac{1}{2}(\beta_5^2+\beta_5-2)m. \tag{3.10}$$

Within a representation where β_5 is diagonal, we choose for example in the *vector* context $\beta_5=\mathrm{diag}(1, 1, 1, -1, -1, -1, 0, 0, 0, 0)$ and equations (3.5) and (3.10) give

$$\begin{aligned}&(P_\mu P^\mu-m^2)\Psi'_i=0 \qquad i=1,2,3\\ &\Psi'_4=\Psi'_5=\ldots=\Psi'_{10}=0.\end{aligned} \tag{3.11}$$

We thus obtain an analogous result to that obtained in the Dirac context (see equation (2.10)) but with $((2s+1)$ for $s=1)$ Klein–Gordon equations admitting Poincaré and products of Poincaré symmetries deduced from the 10 transformed operators

$$V_{\mathrm{K}}^{-1}L_{\mu\nu}V_{\mathrm{K}} \tag{3.12}$$

and

$$V_{\mathrm{K}}^{-1}P_\mu V_{\mathrm{K}}. \tag{3.13}$$

First-order generators are then immediately obtained: there are 26 of them, which can once again be collected in the compact forms

$$W_\mu^{\mathrm{K}} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \qquad M^{\nu\rho} = L^{\nu\rho} + S_\kappa^{\nu\rho} \tag{3.14a}$$

$$W_{\mu\nu}^{\mathrm{K}} = \frac{1}{m}(P_\mu W_\nu^{\mathrm{K}} - P_\nu W_\mu^{\mathrm{K}}) \tag{3.14b}$$

$$A_\mu^{\mathrm{K}} = \frac{1}{m}(\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} W_{\mathrm{K}}^\sigma - \tfrac{1}{2}P_\mu) \tag{3.14c}$$

$$B^{\mathrm{K}} = \tfrac{1}{4}\varepsilon_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma} \tag{3.14d}$$

in correspondence with equations (2.12) in the Dirac case and evidently supplemented by the identity and the 10 Poincaré operators $M_{\mu\nu}$ and P_μ. Once again we recover here symmetry operators belonging to the Poincaré enveloping algebra as already shown in the Dirac context.

3.2. Towards Lie Parasuperalgebras

It is remarkable to now be able to put forward new Lie structures through these Kemmer symmetry operators. By considering once again the 10 second-order translation operators $P_\mu P_\nu$ for evident reasons connected in particular with the Klein–Gordon operator, we can collect together the 31 operators

$$\{I, P_\mu, M_{\mu\nu}, P_\mu P_\nu, W_\mu^{\mathrm{K}}, W_{\mu\nu}^{\mathrm{K}}\} \tag{3.15}$$

in analogy to the set (2.13) in the Dirac context. On the one hand, the set (3.15) does *not* lead to a Lie superalgebra under the Lie superbracket referring to commutators *and* anticommutators (as in the Dirac case): it is easy to show for example that the anticommutators between two $W_{\mu\nu}^{\mathrm{K}}$s do not close. On the other hand, the same set (3.15) leads to a Lie parasuperalgebra under the Lie bracket defined by double commutators [21] or by other specific laws [31]. For this second possibility, note that it is necessary to consider I, P_μ, $M_{\mu\nu}$ and $P_\mu P_\nu$ as even operators and W_μ^{K} and $W_{\mu\nu}^{\mathrm{K}}$ as odd ones in accordance with the characteristics pointed out in the Dirac context.

It should be noticed in general that for each property in the Dirac theory connected with the supersymmetric theory there is a corresponding parallel property in the Kemmer theory connected with the parasupersymmetric theory. The above correspondence between the sets (2.13) and (3.15) is only an illustration of this fact up to the replacement of Lie superstructures by Lie parasuperstructures for example. The same is true at the level of more recent properties obtained in the Dirac theory as hidden supersymmetries [12], so that here we could point out hidden parasupersymmetries in the Kemmer theory. For brevity we shall not proceed in this direction at the moment, but we could define parasupercharges from linear combinations such as $W_i \pm \frac{1}{2}\varepsilon_{ijk} W_{jk}$, etc.

3.3. The particular case of scalar mesons

Up to equation (3.10) in section 3.1 we have not been obliged to select the description of vector or scalar mesons. Formally speaking, the situation is completely parallel for spins 0 and 1 except that, for the spin-0 case within a five-dimensional representation of the Kemmer matrices, we know [32] that the matrix β_5 is identically zero, so that our invertible operators V_K and W_K reduce trivially to the identity. An elegant way to avoid this difficulty is realized by going to an equivalent context where the fifth matrix β_5 can play its role. Let us consider a K(5) Kemmer algebra characterized by the structure relations

$$\beta_A\beta_B\beta_C+\beta_C\beta_B\beta_A=g_{AB}\beta_C+g_{CB}\beta_A \qquad (A, B, C=0,1,2,3,5) \tag{3.16}$$

with $g_{55}=+1$. It contains 462 linearly independent elements [33]. Five of these belong to the centre, and lead to the decomposition

$$462=(1)^2+(6)^2+(10)^2+(10)^2+(15)^2. \tag{3.17}$$

We now propose to exploit the six-dimensional representation to describe scalar mesons. A specific choice for the corresponding matrices is given by

$$\beta_0=e_{1,2}+e_{2,1} \qquad \beta_k=\mathrm{i}(e_{1,k+2}+e_{k+2,1}) \qquad \beta_5=e_{1,6}+e_{6,1} \tag{3.18}$$

where $k=1, 2, 3$ and the symbols $e_{m,n}$ refer to a 6×6 matrix whose non-zero element located at the intersection of the mth row and nth column is equal to one. It is easy to convince ourselves that such a representation is associated with the spin-0 context by evaluating the spin tensor components (3.4) and by showing that they act trivially on the non-redundant components. Within such a representation, the matrix β_5 can be diagonalized by

$$U=U^\dagger=\frac{1}{\sqrt{2}}(e_{1,1}+e_{2,2}+e_{3,3}-e_{4,4}-e_{5,5}-e_{6,6}+e_{1,6}+e_{2,5}+e_{3,4}+e_{4,3}+e_{5,2}+e_{6,1})$$

so that we get

$$\beta_5'=\mathrm{diag}(1,0,0,0,0,-1). \tag{3.19}$$

At this stage, the developments (3.5)–(3.8) apply and the result (3.10) leads to the following system:

$$(P_\mu P^\mu-m^2)\Psi_1''=0 \tag{3.20}$$

$$\Psi_2''=\Psi_3''=\ldots=\Psi_6''=0 \tag{3.21}$$

which is equivalent to equation (3.5) in this spin-0 case. We thus get here ($(2s+1)$ for $s=0$) one Klein–Gordon equation when considering the three spins 0, $\frac{1}{2}$ and 1 in a unique discussion that is summarized in section 4.

4. General comments and conclusions

As a consequence of the previous sections, the different spin cases show analogous properties as far as $s=0, \frac{1}{2}$ or 1 values are considered for *first-order* symmetry operators. Due to the analogous role played by the fifth matrix (γ_5 in the spin-$\frac{1}{2}$ case and β_5 in the spin-0 or spin-1 cases), *we obtained in each case a theory described by wavefunctions without* redundant components satisfying the Klein–Gordon equation as expected. For each sign of the energy we have obtained the required number $(2s+1)$ of such non-trivial components. From the symmetry point of view, we have also deduced 'parallel' conclusions within the Dirac-supersymmetry and Kemmer-parasupersymmetry associations, respectively. One of the main points is that we have collected the extended symmetries in identical formulae (see equations (2.12) and (3.14) for example), showing that all these extended symmetries are simply the resulting products of Poincaré symmetries, so that we are always considering elements of the Poincaré enveloping algebra. By dealing with *symmetric* ($G \equiv \{g_{\mu\nu}\}$) and *antisymmetric* ($\{\varepsilon_{\mu\nu\rho\sigma}\}$) tensors and with only the 10 *first-order* Poincaré generators P_μ and $M_{\mu\nu}$, we immediately get the only scalar (B), 4-vectors (W_μ and A_μ) and second-order tensor ($W_{\mu\nu}$) compactifying our 15 extended operators due to trivial 'identities' such as

$$\varepsilon_{\mu\nu\rho\sigma}L^{\nu\rho}P^{\sigma}=0 \qquad \varepsilon_{\mu\nu\rho\sigma}L^{\mu\nu}L^{\rho\sigma}=0 \qquad P_\mu W^\mu = 0$$

as well as to the characteristic properties of the two Poincaré Casimir operators. These properties explain the following expressions of these 15 operators independently of the particular spin context under study:

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^{\sigma} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}S^{\nu\rho}P^{\sigma} \tag{4.1a}$$

$$W_{\mu\nu} = \frac{1}{m}(P_\nu W_\mu - P_\mu W_\nu) = \frac{1}{2m}(\varepsilon_{\mu\lambda\rho\sigma}P_\nu - \varepsilon_{\nu\lambda\rho\sigma}P_\mu)P^{\sigma}S^{\lambda\rho} \tag{4.1b}$$

$$A_\mu = \frac{1}{m}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}W^{\sigma} = \frac{1}{m}(L^{\alpha\beta}S_{\alpha\beta}P_\mu + 2L^{\alpha\beta}S_{\mu\alpha}P_\beta + S^{\alpha\beta}S_{\alpha\beta}P_\mu + 2S^{\alpha\beta}S_{\mu\alpha}P_\beta) \tag{4.1c}$$

$$B = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}L^{\mu\nu}S^{\rho\sigma} + \tfrac{1}{4}\varepsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}. \tag{4.1d}$$

The last expressions are given in terms of the spin tensor $\{S_{\mu\nu}\}$, ensuring that they act trivially for scalar mesons in particular.

Let us now conclude by adding the two following remarks. First, we notice that, through the above expressions, it is evident that the 15 operators (4.1) are Lie extended symmetry operators for *arbitrary* spin values. It has then to be noticed that they are all *first-order* generators for the spin values $s=0, \frac{1}{2}$ and 1, as can be verified explicitly. For other spin values, the symmetries $W_{\mu\nu}$ and A_μ are, in general, second order. The second remark concerns the general symmetries which are admitted by relativistic covariant equations such as the Bhabha equation [34] for arbitrary spins. We know that there exist [35] Poincaré generators satisfying the Poincaré commutation relations and, consequently, that for arbitrary spins the operators (4.1) exist and belong to the Poincaré enveloping algebra. It can be shown that, except for the cases of spins $0, \frac{1}{2}$ and 1, there are also symmetry operators of the Bhabha equation which do not belong to this enveloping algebra; however, we do not want to go further in this direction in the present context.

References

[1] Wigner E P 1939 *Ann. Math.* **40** 149

[2] Shirokov Iu M 1958 *Sov. Phys.-JETP* **6** 644, 919, 929

[3] Gasiorowicz S 1966 *Elementary Particle Physics* (New York: Wiley)

[4] Bogolubov N N, Logunov A A and Todorov I T 1975 *Introduction to Axiomatic Quantum Field Theory* (Reading, MA: Benjamin)

[5] Cornwell J F 1984 *Group Theory in Physics* vols I, II (London: Academic)

[6] Fushchich W I and Nikitin A G 1987 *Symmetries of Maxwell's Equations* (Dordrecht: Reidel)

[7] Lie S 1893 *Transformationgruppen* (Leipzig: Teubner)
Lie S and Engel F 1893 *Theorie der Transformationgruppen* (Leipzig: Teubner)

[8] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)

[9] Fushchich W I and Nikitin A G 1987 *J. Phys. A: Math. Gen.* A **20** 537

[10] Beckers J and Debergh N 1991 *J. Phys. A: Math. Gen.* A **24** 2889
[11] Kac V 1977 *Adv. Math.* **26** 8
[12] Beckers J, Debergh N and Nikitin A G 1992 *Phys. Lett.* **279B** 333
[13] Witten E 1981 *Nucl. Phys.* B **188** 513
[14] Jackiw R 1984 *Phys. Rev.* D **29** 2375
Hughes R J, Kostelecky V A and Nieto M M 1986 *Phys. Rev.* D **34** 1100
[15] Beckers J and Debergh N 1990 *Phys. Rev.* D **42** 1255
[16] Duffin R J 1938 *Phys. Rev.* **54** 1114
Kemmer N 1939, *Proc. R. Soc.* A **173** 91
Petiau G 1936 *Acad. R. Belgique, Classe Sci., Mémoires Coll. 8* **16** no. 2
[17] Umezawa H 1956 *Quantum Field Theory* (Amsterdam: North-Holland)
[18] Rubakov V A and Spiridonov V P 1988 *Mod. Phys. Lett.* A **3** 1337
[19] Beckers J and Debergh N 1990 *Nucl. Phys.* B **340** 767
[20] Beckers J, Debergh N and Nikitin A G 1992 *J. Math. Phys.* in press
[21] Beckers J and Debergh N 1990 *J. Phys. A: Math. Gen.* **23** L751S; 1991 *J. Phys. A: Math. Gen.* **24** L597
[22] Bargmann V and Wigner E P 1948 *Proc. Natl. Acad. Sci. USA* **34** 211
[23] Hilgevoord J and Wouthuysen S A 1963 *Nucl. Phys.* **40** 1
[24] Beckers J 1967 *Bull. Soc. R. Sci. Liège* **36** 643
[25] Fushchich W I and Nikitin A G 1977 *Lett. Nuovo Cimento* **19** 347
[26] Lubanski J K 1941 *Physica* **8** 44
[27] Rose M E 1961 *Relativistic Electron Theory* (New York: Wiley)
[28] Nikitin A G 1991 *Ukr. Mat. Zurn.* **43** 1388
[29] Spiridonov V P 1991 *J. Phys. A: Math. Gen.* **24** L529
[30] Korchenmsky G P 1991 *Phys. Lett.* **267B** 597
[31] Durand S and Vinet L 1990 *J. Phys. A: Math. Gen.* **23** 3661
[32] Harish-Chandra 1946 *Proc. R. Soc.* A **186** 502
[33] Garnir H 1949 *Acad. R. Belgique, Classe Sci., Mémoires Coll. 8* **23** no. 9
Lubanski J K and Rosenfeld L 1942 *Physica* **9** 117
[34] Bhabha H J 1945 *Rev. Mod. Phys.* **17** 300; 1949 *Rev. Mod. Phys.* **21** 451
[35] Krajcik R A and Nieto M M 1975 *Phys. Rev.* D **11** 1459